PERGAMON

International Journal of Solids and Structures 40 (2003) 5753–5767

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

The non-homogeneous biharmonic plate equation: fundamental solutions

George D. Manolis [a,*], Tsviatko V. Rangelov [b], Richard P. Shaw [c]

[a] Department of Civil Engineering, Aristotle University, Panepistimioupolis, P.O. Box 502, Thessaloniki GR-54006, Greece
[b] Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia 1113, Bulgaria
[c] Department of Civil Engineering, State University of New York, Buffalo, New York 14260, USA

Received 8 February 2003; received in revised form 12 June 2003

Abstract

Real materials and structural components are often non-homogeneous, either by design or because of the physical composition and imperfections in the underlying material. Thus, analytical solutions for non-homogeneous materials under mechanical loads are of considerable interest to engineers and have widespread applications, given the prevalence of these materials in fields as diverse as aerospace, construction, electronics, etc. More precisely, those are essentially composites with carefully manufactured properties that yield desirable mechanical characteristics and properties, such as optimal arrangement of the material, minimum weight, etc. To this end, the displacement fundamental solution (or Green's function) corresponding to a point force for the non-homogenous biharmonic equation in two dimensions are derived in this work by employing a conformal mapping technique in conjunction with the Radon transformation. These functions, besides being useful in their own right, can also be used within the context of integral equation formulations for the solution of boundary-value problems. Finally, a series of numerical examples that deal with the non-homogeneous plate on elastic foundation problem serve to illustrate the present method.

Keywords: Biharmonic operator; Conformal mapping; Elastic sub-grade; Green's functions; Non-homogeneous media; Plates; Radon transformation

1. Introduction

Plates on elastic foundations date back to Hertz (1895), who investigated the case of a floating thin plate. Since then, a number of analytical solutions have followed, as for instance plates resting on an elastic half-space (Schultze and Mahs, 1950). A comprehensive account of these closed-form solutions reaching into the 1960's can be found in the treatise by Timoshenko and Woinowsky-Krieger (1970). Gradually, emphasis has shifted to numerical methods of solution, primarily through development of various types of finite elements for representing plates and shells (Zienkiewicz and Taylor, 2000). As far as boundary integral

* Corresponding author. Tel.: +3-2310-995663; fax: +3-2310-995769.
E-mail address: gdm@civil.auth.gr (G.D. Manolis).

0020-7683/$ - see front matter
doi:10.1016/S0020-7683(03)00343-3

equation methods are concerned, we have the pioneering work of Jawson and Maiti (1968), who introduced a complete integral formulation for thin plates using as kernel functions the singular solutions for a point load on a circular plate obeying the radiation boundary condition. Following that seminal paper, as well as early papers by Bezine (1978) and Stern (1979), much work has gone into developing what is now known as the boundary element method (BEM) for problems involving plates. Besides the classical bending of thin plates of various shapes, boundary conditions and loading, we have work on orthotropic (Irshchik, 1984), anisotropic (Heng and McCammond, 1992) and thick plates (Benitez, 1991), plates on elastic foundations (Katsikadelis and Armenakas, 1984; Bezine, 1988) and plates of variable thickness (Katsikadelis and Sapountzakis, 1991), plate stability (Kamiya et al., 1984), plate dynamics (Leissa, 1969), nonlinear plates (Song and Mukherjee, 1989; O'Donoghue and Atluri, 1987) and finally contact problems involving plates (Sapountzakis and Katsikadelis, 1992; Faruque and Zaman, 1991). We mention here in passing that a state-of the art review (Providakis and Beskos, 1999) on just plate dynamics by the BEM, written a few years ago, referenced about 150 papers.

The need remains, however, for specialized fundamental solutions that will enhance the capabilities of the BEM to treat plates made of non-homogenous materials. To that end, the present paper focuses on development of Green's functions for point loads using conformal mapping in conjunction with the Radon transformation. The specific type of inhomogeneity is dependent on the type of mapping prescribed; for instance, an exponential mapping yields a plate modulus that also varies exponentially with distance from the source of application of the load, a quadratic mapping gives a modulus that varies as the radial distance raise to the second power from the source, etc. Conformal mapping methods for obtaining fundamental solutions have been introduced rather recently (Shaw and Gipson, 1995; Shaw and Manolis, 2000) as an alternative to integral transforms (Fourier, Laplace, Hankel, etc.), which despite their generality leave an inverse transformation in the form of a contour integral over the complex plane as the final step. It is worth mentioning that the inverse transformation when using conformal mapping is simple and, in any case, no more difficult than the direct transformation. The drawback is that we are dealing with an 'inverse' method, i.e., the material profile recovered depends on the conformal mapping used and cannot be established independently of it.

The second step in the solution procedure involves use of the Radon transform (Burkovics, 1994; Schmetterer, 1994) for computing a fundamental solution of the 'mapped' biharmonic equation. With the Radon transform, a function of two variables is reconstructed from its integrals over all straight lines in the plane or from contour integrals over smooth curves in 2D. In the most general form of the transformation, a function of n variables is reconstructed from its integrals over all n hyper-planes. We mention in passing that the Radon transformation went unnoticed for many years and was rediscovered in the 1950's in conjunction with medical imaging that lead to development of the computer-assisted tomography (CAT-scan) technique. Finally, a series of examples serve to illustrate the present methodology and to contrast the fundamental solutions obtained herein with some of the classical ones involving a floating plate and a plate on elastic half-space described by the Boussinesq influence function.

Solutions to the biharmonic equation with non-constant coefficients go beyond plate theory. For instance, thin shell analysis usually proceeds along two paths, starting with the breakdown of the shell governing equations (Novozhilov, 1964) into either two fourth-order differential equations for the deflection and stress function or starting with the usual plate bending equation supplemented by membrane action. In either case, information provided in this work may prove to be useful in establishing integral equation formulations for shallow shells.

2. Methodology

The non-homogeneous biharmonic operator is defined in a Cartesian coordinate system Ox_1x_2 and for a simply-connected domain $\Omega \subset R^2$ as a fourth-order, linear differential operator

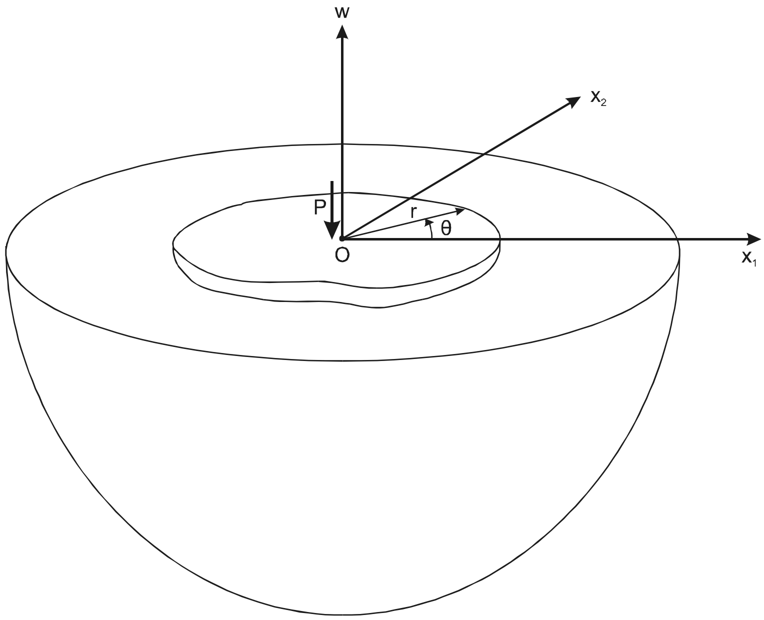

Fig. 1. Non-homogeneous plate resting on an elastic subgrade.

$$L(x, \partial_x)u(x) \equiv \Delta_x(K(x)\Delta_x + \beta)u(x) \tag{1}$$

In the above, $x = (x_1, x_2)$, $\Delta_x = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the $2D$ Laplacian, function $K(x) \in C^\infty(\Omega)$ such that $K_a \leqslant K(x) \leqslant K_b$ where K_a, K_b are positive constants and finally $\beta \geqslant 0$ is a constant. From a physical viewpoint, $u(x)$ is the plate deflection, $K(x)$ the elastic modulus and β the reaction of a sub-grade material (see Fig. 1).

Since our final objective is to solve well-posed boundary value problems (BVP) for L in Ω by the BEM, the first step here is to recover an efficient (for numerical implementation purposes) fundamental solution for operator L. Specifically, we seek Green's function $U(x, \xi)$ such that

$$L(x, \partial_x)U(x, \xi) = -\delta(x, \xi) \tag{2}$$

where δ is Dirac delta (generalized) function and $\xi = (\xi_1, \xi_2)$ is the source point where a unit load is applied. It is well known (John, 1955) that if $K(x)$ is analytical (or C^∞) in Ω, then fundamental solution U exists. This is also true for many other operators, for which $K(x)$ (or an equivalent function) satisfies weaker smoothness requirements. If $K(x) = K_0$ (a constant), then U can be obtained by the Fourier transform or by other methods (such as power series expansions) and its form involves a combination of Bessel functions (Timoshenko and Woinowsky-Krieger, 1970). Such solutions have been used in the past for solving BVP with the BEM (see Brebbia et al., 1984). Our aim is to obtain a fundamental solution of Eq. (1) for the class of admissible functions $K(x)$ defined below.

2.1. Conformal mapping

Let Y be a conformal mapping (Sidorov et al., 1982), defined in Ω through reference to a regular function $Y(z) = y_1(x) + iy_2(x)$, $z = x_1 + ix_2$. Denote by $J(x) = \det(\partial y/\partial x)$ the Jacobian (determinant) of the transformation $y = y(x)$, and by $J_1 = J^{-1}$ the Jacobian of the inverse transform $x = x(y)$. In view of the Cauchy–Riemann conditions

$$\frac{\partial y_1}{\partial x_1} = \frac{\partial y_2}{\partial x_2}, \quad \frac{\partial y_1}{\partial x_2} = -\frac{\partial y_2}{\partial x_1} \tag{3}$$

functions $y_1(x)$, $y_2(x)$ are harmonic in Ω, i.e., satisfy Laplace's equation $\Delta_x y_j(x) = 0$, $j = 1, 2$. Furthermore,

$$J(x) = \left(\frac{\partial y_1}{\partial x_1}\right)^2 + \left(\frac{\partial y_1}{\partial x_2}\right)^2 \neq 0, \quad \text{in } \Omega, \quad J_1(y) = \left(\frac{\partial x_1}{\partial y_1}\right)^2 + \left(\frac{\partial x_1}{\partial y_2}\right)^2 \neq 0 \quad \text{in } Y(\Omega) \tag{4}$$

Under the change of variables to Y, Laplacian Δ_x in Ω is transformed to an operator $J_1(y)\Delta_y$ in $Y(\Omega)$, which is essentially a scaled Laplacian (Shaw and Gipson, 1995; Shaw and Manolis, 2000). Specifically,

$$\frac{\partial \cdot}{\partial x_j^2} = \left(\frac{\partial y_1}{\partial x_j}\right)^2 \frac{\partial^2 \cdot}{\partial y_1^2} + \left(\frac{\partial y_2}{\partial x_j}\right)^2 \frac{\partial^2 \cdot}{\partial y_1^2} + 2\left(\frac{\partial y_1}{\partial x_j}\frac{\partial y_2}{\partial x_j}\right)\frac{\partial^2 \cdot}{\partial y_1 \partial y_2} + \frac{\partial^2 y_1}{\partial x_j}\frac{\partial \cdot}{\partial y_1} + \frac{\partial^2 y_2}{\partial x_j}\frac{\partial \cdot}{\partial y_2} \tag{5}$$

where $\cdot$ indicates the argument. Using Eqs. (4) and (5) yields

$$\Delta_x \cdot = \left[\left(\frac{\partial y_1}{\partial x_1}\right)^2 + \left(\frac{\partial y_1}{\partial x_2}\right)^2\right]\left(\frac{\partial^2 \cdot}{\partial y_1^2} + \frac{\partial^2 \cdot}{\partial y_2^2}\right) + 2\left(\frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_1} + \frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_2}\right)\frac{\partial^2 \cdot}{\partial y_1 \partial y_2} + (\Delta_x y_1)\frac{\partial \cdot}{\partial y_1} + (\Delta_x y_2)\frac{\partial \cdot}{\partial y_2} \tag{6}$$

and it follows from Eq. (3) that

$$\Delta_x \cdot = J_1(y)\Delta_y \cdot \tag{7}$$

Define as $K_\Omega = \{K(x), x \in \Omega\}$ the set of $C^\infty(\Omega)$-continuous functions such that condition $K(x)/J(x) = \gamma$ (with $\gamma > 0$) is fulfilled in $Y(\Omega)$, provided conformal mapping Y exists. The equivalent statement in the new (mapped) coordinates is

$$K_1(y)J_1(y) = \gamma \tag{8}$$

where $K_1(y) = K(x(y))$. The set K_Ω can now be characterized as follows.

If $y_1(x)$ is a harmonic function in Ω, then there exists a unique (to within a constant) harmonic function $y_2(x)$ in Ω, namely the conjugate to $y_1(x)$, such that function $Y = y_1(x) + \mathrm{i}y_2(x)$ satisfies the Cauchy–Riemann conditions (Sidorov et al., 1982) given in Eq. (3). The Jacobian of the conformal mapping defined by Y is determined from Eq. (4). Using Eqs. (3) and (4), it follows that function $y_1(x)$ in Ω must satisfies the conditions

$$\begin{cases} \Delta_x y_1(x) = 0 \\ \left(\frac{\partial y_1}{\partial x_1}\right)^2 + \left(\frac{\partial y_1}{\partial x_2}\right)^2 = \frac{1}{K(x)} \end{cases} \tag{9}$$

Therefore, function $K(x) \in K_\Omega$ if and only if function $y_1(x)$ exists in Ω and satisfies Eq. (9). As examples of such transformations (see also the Appendix A) we have the following cases:

(a) Exponential, where $y_1(x) = \mathrm{e}^{x_1}\sin x_2$; then $K(x) = \mathrm{e}^{-2x_1}$.
(b) Quadratic, where $y_1(x) = x_1x_2$; then $K(x) = [(x_1^2 + x_2^2)]^{-1}$.
(c) Power law, where $y_1(x) = x_1x_2 + a_1x_1 + b_1 + a_2x_2 + b_2$; then $K(x) = \left((x_1 + a_1)^2 + (x_2 + a_2)\right)^{-1}$ in the domain $\Omega \subset R^2 \setminus \{-a_1, -a_2\}$.

In sum, the system of Eqs. (9) for determining $y_1(x)$ when $K(x)$ is a given bounded, positive and smooth function in a simply connected domain Ω is over-determined and, generally speaking, a solution does not exist. In this work, we will recover a fundamental solution for Eq. (2) only when function $K(x) \in K_\Omega$, since a detailed study of Eq. (9), including complete description of set K_Ω, is in itself a separate task.

2.2. Radon transformation

Let $K(x) \in K_\Omega$ and Y be the corresponding conformal mapping with $J(x)$ its Jacobian. Then, by employing Eqs. (7) and (8), biharmonic Eq. (2) is transformed in $\Omega_1 = Y(\Omega)$ as follows:

$$\left[\gamma J_1(y)\Delta_y(\Delta_y + \beta/\gamma)\right]V(y,\xi) = -\delta(y,\xi) \tag{10}$$

Since for a point function, $(1/(\gamma J_1(y)))\delta(y,\xi) = (1/(\gamma J_1(\xi)))\delta(y,\xi)$, Eq. (10) can be re-written (with $\beta_1 = \beta/\gamma$) as

$$\Delta_y(\Delta_y + \beta_1)V(y,\xi) = -(1/(\gamma J_1(\xi)))\delta(y,\xi) \tag{11}$$

We will now find a fundamental solution of Eq. (11) using the Radon transformation in 2*D*. More specifically, the Radon transformation for a function $f \in \Im$ (the class of rapidly decreasing functions in C^∞) is defined (Ludwig, 1966; Zayed, 1996) as

$$\mathrm{R}(f) = \hat{f}(s,n) = \int_{\langle y,n\rangle = s} f(y)\,\mathrm{d}S = \int f(y)\delta(s - \langle y,n\rangle)\,\mathrm{d}y, \quad s \in R^1,\ n \in S^1 \tag{12}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in R^2. Also, the inverse Radon transformation is

$$f(y) = \mathrm{R}^{-1}(\hat{f}(s,n)) = \frac{1}{4\pi^2}\int_{|n|=1}\left(\int_{-\infty}^{+\infty}\frac{\partial_p f(p,n)}{s-p}\,\mathrm{d}p\right)\mathrm{d}n \tag{13}$$

A useful property of the transformation is linearity, i.e.,

$$\mathrm{R}(af(y) + bg(y)) = a\mathrm{R}(f(y)) + b\mathrm{R}(g(y)) \tag{14}$$

Furthermore, if $L(\partial_y)$ is a homogeneous differential operator of degree k with constant coefficients, then

$$\mathrm{R}(L(\partial_y)f(y)) = L(n)\frac{\partial^k}{\partial s^k}\hat{f}(s,n) \tag{15}$$

The Radon transform can also be defined in the space of distributions (see Gelfand et al., 1966) as

$$\mathrm{R}(\delta(y,\xi)) = \delta(s - \langle n,\xi\rangle) \tag{16}$$

Applying the Radon transform to both sides of Eq. (11) and using the above properties, which give $\mathrm{R}(\Delta_y f) = (n_1^2 + n_2^2)\partial_s^2\hat{f} = \partial_s^2\hat{f}$, yields the transformed biharmonic equation as

$$\partial_s^2(\partial_s^2 + \beta_1)\widehat{V}(s,n) = -\frac{1}{\gamma J_1(\xi)}\delta(s - \langle n,\xi\rangle) \tag{17}$$

The solution that follows will first consider the possibility $\beta_1 > 0$, while the special case $\beta_1 = 0$ will be treated later.

We follow Vladimirov (1984) and solve the equivalent initial-value problem for the ordinary, transformed differential equation

$$\begin{cases} d_s^2(d_s^2 + \beta_1)Z(s) = 0 \\ Z(\tau) = Z'(\tau) = Z''(\tau) = 0, Z'''(\tau) = -(1/(\gamma J_1(\xi))) \end{cases} \tag{18}$$

with $\tau = \langle n, \xi \rangle$. The unique solution of Eq. (18) is $Z(s) = a + bs + c\mathrm{e}^{\mathrm{i}\sqrt{\beta_1}s} + d\mathrm{e}^{-\mathrm{i}\sqrt{\beta_1}s}$, where coefficients

$$a = -\frac{\tau}{\beta_1 \gamma J_1(\xi)}, \quad b = \frac{1}{\beta_1 \gamma J_1(\xi)}, \quad c = \frac{\mathrm{e}^{-\mathrm{i}\sqrt{\beta_1}\tau}}{2\mathrm{i}\beta_1^{3/2}\gamma J_1(\xi)}, \quad d = -\frac{\mathrm{e}^{\mathrm{i}\sqrt{\beta_1}\tau}}{2\mathrm{i}\beta_1^{3/2}\gamma J_1(\xi)}$$

Then, the solution to Eq. (17) is

$$\widehat{V}(s,n) = H(s-\tau)Z(s) = \frac{H(s-\tau)}{\beta J_1(\xi)}\left[(s-\tau) + \frac{1}{\beta_1^{1/2}}\sin[\sqrt{\beta_1}(s-\tau)]\right] \tag{19}$$

where $H(\cdot)$ is the Heaviside (or step) function.

2.3. Green's function

The next step is to apply the inverse Radon transform defined in Eq. (13) to recover the fundamental solution as

$$V(y,\xi) = -\frac{1}{4\pi^2 \beta J_1(\xi)} \int_{|n|=1} \left[\ln\eta - \left[ci(\sqrt{\beta_1}\eta)\cos(\sqrt{\beta_1}\eta) + si(\sqrt{\beta_1}\eta)\sin(\sqrt{\beta_1}\eta)\right]\right]\Big|_{\eta=|\langle n, y-\xi\rangle|} \mathrm{d}n \tag{20}$$

where

$$ci(z) = -\int_z^{+\infty} \frac{\cos t}{t}\,\mathrm{d}t, \quad si(z) = -\int_z^{+\infty} \frac{\sin t}{t}\,\mathrm{d}t$$

are the cosine and sine integrals, respectively. If we denote

$$V(y,\xi) = -\frac{1}{4\pi^2 \beta J_1(\xi)} \int_{|n|=1} (\widetilde{V}_1 + \widetilde{V}_2)\big|_{s=\langle n, y-\xi\rangle} \mathrm{d}n,$$

where

$$\widetilde{V}_1 = \int_{-\infty}^{+\infty} \frac{\partial_\sigma[H(\sigma-\tau)(\sigma-\tau)]}{s-\sigma}\,\mathrm{d}\sigma \quad \text{and} \quad \widetilde{V}_2 = \frac{1}{\beta_1^{1/2}} \int_{-\infty}^{+\infty} \frac{\partial_\sigma[H(\sigma-\tau)\sin(\sqrt{\beta_1}(\sigma-\tau))]}{s-\sigma}\,\mathrm{d}\sigma,$$

then by using the properties of the Radon transformation, and of the delta and Heaviside functions, we recover the following results in the sense of distributions:

$$\widetilde{V}_1 = \int_{-\infty}^{+\infty} \frac{\delta(\sigma-\tau)(\sigma-\tau) + H(\sigma-\tau)}{s-\sigma}\,\mathrm{d}\sigma = \int_{\tau}^{+\infty} \frac{\mathrm{d}\sigma}{s-\sigma} = \ln|s-\sigma|\big|_{\tau}^{+\infty} = -\ln|s-\tau|,$$

$$\begin{aligned}\widetilde{V}_2 &= \frac{1}{\beta_1^{1/2}} \int_{-\infty}^{+\infty} \frac{\delta(\sigma-\tau)\sin(\sqrt{\beta_1}(\sigma-\tau)) + H(\sigma-\tau)\cos(\sqrt{\beta_1}(\sigma-\tau))}{s-\sigma}\,\mathrm{d}\sigma \\ &= \int_{\tau}^{+\infty} \frac{\cos(\sqrt{\beta_1}(\sigma-\tau))}{s-\sigma}\,\mathrm{d}\sigma = -\int_{\tau-s}^{+\infty} \frac{\cos(\sqrt{\beta_1}(p+s-\tau))}{p}\,\mathrm{d}p\end{aligned}$$

In the above, $p = s - \sigma$. Furthermore,

$$\widetilde{V}_2 = -\int_{\tau-s}^{+\infty} \frac{\cos\sqrt{\beta_1}p}{p}\,\mathrm{d}p \cdot \cos(\sqrt{\beta_1}(\tau-s)) - \int_{\tau-s}^{+\infty} \frac{\sin\sqrt{\beta_1}p}{p}\,\mathrm{d}p \cdot \sin(\sqrt{\beta_1}(\tau-s)),$$

and

$$\widetilde{V}_2 = -\int_{\sqrt{\beta_1}(\tau-s)}^{+\infty} \frac{\cos q}{q}\,\mathrm{d}q \cdot \cos(\sqrt{\beta_1}(\tau-s)) - \int_{\sqrt{\beta_1}(\tau-s)}^{+\infty} \frac{\sin q}{q}\,\mathrm{d}q \cdot \sin(\sqrt{\beta_1}(\tau-s)),$$

where $\sqrt{\beta_1}p = q$. Finally,

$$\widetilde{V}_2 = c\mathrm{i}(\sqrt{\beta_1}(\tau-s))\cos(\sqrt{\beta_1}(\tau-s)) + s\mathrm{i}(\sqrt{\beta_1}(\tau-s))\sin(\sqrt{\beta_1}(\tau-s)).$$

In order to check that Eq. (20) is indeed a fundamental solution of Eq. (11), we use calculus of distributions and an integral representation (Vladimirov, 1984) of the delta function over a unit circle in 2D.

$$\delta(z) = \frac{1}{4\pi^2}\int_{|n|=1} \frac{1}{\langle n,z\rangle^2}\,\mathrm{d}n \tag{21}$$

Finally, the fundamental solution of Eq. (2) in terms of the x variables is $U(x,\xi) = V(y(x),\xi)$. Its first spatial derivatives are obtained as follows:

$$\partial_{x_j} U(x,\xi) = \partial_{y_k} V(y,\xi)\frac{\partial y_k}{\partial x_j}$$

where

$$\partial_{y_j} V(y,\xi) = -\frac{\sqrt{\beta_1}}{4\pi^2\beta J_1(\xi)}\int_{|n|=1}\left[\mathrm{ci}(\sqrt{\beta_1}\eta)\sin(\sqrt{\beta_1}\eta) - s\mathrm{i}(\sqrt{\beta_1}\eta)\cos(\sqrt{\beta_1}\eta)\right]\Big|_{\eta=|\langle n,y-\xi\rangle|} n_k \mathrm{sgn}(n,y-\xi)\,\mathrm{d}n \tag{22}$$

The remaining derivatives of $U(x,\xi)$ up to third order, that would also be required within the context of a BEM formulation, are obtained in a similar way as above. Back-substitution of these results satisfies Eq. (11) identically. We note the difference between the classical fundamental solution for a homogeneous plate, where modulus $K(x) = K_0$, and the above solution in which $U(x,\xi)$ is no longer a function of the relative distance $(x-\xi)$ between source and receiver, but depends on both x and ξ separately in a more complicated way.

2.4. Freely-supported non-homogeneous plate

This case is recovered when $\beta = 0$ in the biharmonic operator of Eq. (1). Specifically, the Radon transformed Eq. (17) attains a simpler form, i.e.,

$$\partial_s^4 \widehat{V}(s,n) = -\frac{1}{\gamma J_1(\xi)}\delta(s-\langle n,\xi\rangle) \tag{23}$$

This equation is again solved as an initial value problem (Vladimirov, 1984)

$$\begin{cases} d_s^4 Z(s) = 0 \\ Z(\tau) = Z'(\tau) = Z''(\tau) = 0, Z'''(\tau) = -\dfrac{1}{\gamma J_1(\xi)} \end{cases} \tag{24}$$

where $\tau = \langle n,\xi\rangle$. The unique solution of Eq. (24) is now $Z(s) = a + bs + cs^2 + \mathrm{d}s^3$, where

$$a = \frac{\tau^3}{6\gamma J_1(\xi)}, \quad b = -\frac{\tau^2}{2\gamma J_1(\xi)}, \quad c = \frac{\tau}{2\gamma J_1(\xi)}, \quad d = -\frac{1}{6\gamma J_1(\xi)}$$

Thus, the complete transformed fundamental solution is

$$\widehat{V}(s,n) = H(s-\tau)Z(s) = -\frac{H(s-\tau)}{6\gamma J_1(\xi)}(s-\tau)^3 \tag{25}$$

Substitution in Eq. (23) yields

$$\partial_s^4 \widehat{V} = -\frac{1}{\gamma J_1(\xi)}\delta(s-\tau)$$

which confirms that the solution is correct. Finally, applying the inverse Radon transform we obtain the fundamental solution

$$V(y,\xi) = -\frac{1}{8\pi^2\gamma J_1(\xi)}\int_{|n|=1}\eta^2\left(\frac{3}{2}-\ln|\eta|\right)\bigg|_{\eta=\langle n,y-\xi\rangle}\mathrm{d}n \tag{26}$$

Denote

$$V(y,\xi) = -\frac{1}{4\pi^2 6\gamma J_1(\xi)}\int_{|n|=1}\widetilde{V}\big|_{\eta=\langle n,y-\xi\rangle}\mathrm{d}n$$

where

$$\widetilde{V} = \int_{-\infty}^{+\infty}\frac{\partial_\sigma[H(\sigma-\tau)(\sigma-\tau)^3]}{s-\sigma}\mathrm{d}\sigma$$

By using the properties of the Radon transform, of the delta function and of Heaviside function we get (in the sense of distributions) that

$$\widetilde{V} = \int_{-\infty}^{+\infty}\frac{\delta(\sigma-\tau)(\sigma-\tau)+3H(\sigma-\tau)(\sigma-\tau)^2}{s-\sigma}\mathrm{d}\sigma = \int_{\tau}^{+\infty}\frac{(\sigma-\tau)^2\,\mathrm{d}\sigma}{s-\sigma}$$

$$\widetilde{V} = -3\int_{s-\tau}^{+\infty}\frac{(s-\tau-p)^2}{p}\mathrm{d}p$$

where $p = s-\sigma$, and finally

$$\widetilde{V} = -3\int_{s-\tau}^{+\infty}\left[\frac{(s-\tau)^2}{p}-2(s-\tau)+p\right]\mathrm{d}p = -3(s-\tau)^2\left(\frac{3}{2}-\ln|s-\tau|\right)$$

which completes the solution.

To check that Eq. (26) is indeed a fundamental solution of the problem, we use calculus of distributions and the representation of the delta function as a line integral over the unit circle for the general case $\beta \neq 0$.

3. The equivalent homogeneous plate

The biharmonic equation governing bending of an isotropic thin plate resting on an elastic foundation under conditions of radial symmetry is

$$D\Delta_r\Delta_r w(r) = q(r) - p(r) \tag{27}$$

where D is the plate modulus, q is the load, p is the subgrade reaction and r is the radial distance between source and receiver. We revert here to the more standard notation used in structural analysis and define as $W(r)$ the fundamental solution of the above equation under a point load at the origin.

(a) *Floating plate:* If the subgrade reaction is simply proportional to the plate's deflection (i.e., $p(r) = kw(r)$), then the solution of Eq. (27) can be recovered (Hertz, 1895) as a power series or in terms of modified Bessel functions. Specifically,

$$W(r) = -(P\ell^2/2\pi D)k\mathrm{ei}(r) \tag{28}$$

where P is the magnitude of the point load applied at the origin of the coordinate system, $k$ei is a Kelvin function and $\ell^4 = D/k$ is a length parameter.

(b) *Plate on elastic subgrade:* For this case, the Boussinesq solution is employed, which gives the compliance of the elastic half-space as $k_0(\alpha) = (1 - \nu_0)^2/(\pi E_0 \alpha)$, with E_0, ν_0 the modulus of elasticity and Poisson's ratio of the halfspace, respectively, while α is the distance projected on the free surface between source and receiver. By defining a new length parameter as $\ell_0^3 = D/k_0$, the solution for the plate deflection (Schultze and Mahs, 1950) is

$$W(r) = \frac{P\ell_0^2}{2\pi D}\int_0^\infty \frac{J_0(\lambda r/\ell_0)\,\mathrm{d}\lambda}{1+\lambda^3} \tag{29}$$

where J_0 is the zero-order Bessel function and $\lambda = \alpha\ell_0$.

(c) *Free plate:* The solution for a point force at the center of a circular plate is well-known and included here for completeness purposes. Specifically, the basic form (Timoshenko and Woinowsky-Krieger, 1970) is

$$W(r) = -(P/8\pi D)r^2\ln(r) \tag{30}$$

Alternative forms, which include zero-displacement boundary conditions and have been used in BEM formulations (Bezine, 1978; Stern, 1979), are $W(r) = -(P/8\pi D)r^2\ln(r/r_0)$ and $W(r) = -(P/16\pi D)r^2(\ln(r) - 1)$.

4. Numerical implementation

The solutions derived using the present methodology were numerically evaluated for the case of an elastic plate with the following reference properties:

$$E = 2.8\times 10^7 \text{ kN m},\quad \nu = 0.25,\quad d = 0.20 \text{ m} \tag{31}$$

where d is the plate's thickness. These values yield a plate stiffness modulus $D = Ed^3/12(1-\nu^2) = 19{,}911$ kN m. The plate rests on elastic subgrade defined by the following constants:

$$E_0 = 2.010^5 \text{ kN m},\quad \nu_0 = 0.35 \tag{32}$$

which yield $k_0 = 113{,}960$ kN/m^2 and $\ell_0 = 0.559$ m. For the floating plate case, $k = k_0$ (but in kN/m^3) and $\ell = 0.646$ m.

Two types of non-homogeneous plates are examined here that are defined by the exponential and quadratic conformal mappings. In the former case,

$$y_1 = \exp(x_1)\cos x_2,\quad y = \exp(x_1)\sin x_2,\quad J = \exp(2x_2),\quad K(x) = \gamma\exp(-2x_1) \tag{33}$$

where (x_1, x_2) are identified with polar coordinates (r, θ), while in the latter case

$$y_1 = x_1^2 - x_2^2,\quad y_2 = 2x_1x_2,\quad J = 4(x_1^2 + x_2^2),\quad K(x) = \gamma/4(x_1^2 + x_2^2) \tag{34}$$

As far as the parameters appearing in conjunction with the governing biharmonic equation (1) are concerned, $\gamma = K_0 = D$ and $\beta = k_1$, where k_1 is now the subgrade reaction that is proportional to the curvature (i.e., the second derivative of the displacement). As was the case with the homogeneous plate, $k_1 = k_0$ (but in kN/m) and $\ell_1 = \sqrt{D/k_1} = 0.418$ m. Finally, $\sqrt{\beta_1} = \sqrt{\beta/\gamma} = 1/l_1$ and the point load at the origin is conveniently taken as $P = 1000$ kN.

First we concentrate on the homogeneous plate; Fig. 2 plots the deflection of both floating plate and plate on elastic subgrade with the Boussinesq solution, as functions of distance from the point of application of

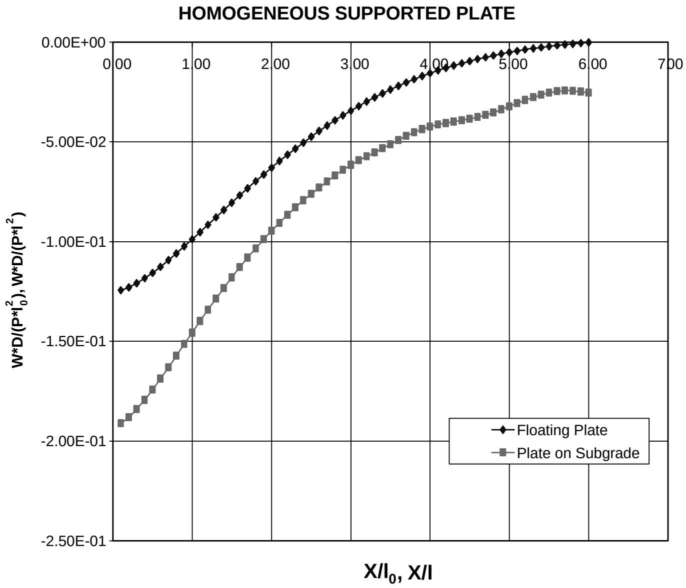

Fig. 2. Normalized deflection versus normalized radial distance for the homogeneous floating plate and the homogeneous plate on elastic subgrade.

the load. All quantities here have been normalized, namely the length by dividing with ℓ_0 (or ℓ) and the deflection by factor $D/P\ell_0^2$ (or $D/P\ell^2$). These graphs are to within plotting accuracy of the results appearing in Timoshenko and Woinowsky-Krieger (1970). We note at this point that the semi-infinite integral appearing in Eq. (29) for the plate on subgrade was numerically evaluated using the special Gauss–Legendre quadrature formula with twenty integration points (Stroud and Secrest, 1966). Fig. 3 is simply the solution for the free plate under the point load, depicting the three possible variations of the fundamental solution discussed in the previous section. Of course, all three graphs are equivalent and have simply been rotated due to imposition of the specific boundary condition. They are shown here because Green's function for the free non-homogeneous plate, Eq. (26), reduces to the plot (Stern, 1979) when the conformal mapping collapses to a one-to-one correspondence between $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

The next series of results are for the non-homogenous plate. Fig. 4 depicts the normalized plate modulus for both exponential and quadratic conformal mapping cases as functions of distance from the source, which is calibrated with respected to length parameter ℓ_1. In terms of notation, $D(= K_0)$ is the reference value of the plate's modulus at the origin. We note that $K(x)$ for the quadratic mapping case is singular at the origin (what we have there is essentially a rigid plate), while $K(x)$ converges to D at $x = 0$ for the exponential mapping case. Furthermore, because of radial symmetry, distance x coincides with radius r in the exponential mapping. For the quadratic mapping, all plots are restricted to lie along the x_1-axis.

Fig. 5 plots the non-homogeneous free plate deflection versus distance form the source. We observe that the deflected shape of the plate defined by the quadratic mapping follows that of the equivalent homogeneous free plate, while the deflections of the plate defined by the exponential mapping increase quite dramatically past a certain distance from the source. This latter behavior is due to the fact that we have a large drop in the plate's stiffness away from the center. In the former case, however, the plate's rigidity is comparable to that of the equivalent homogeneous plate for the key area surrounding the origin. Numerical integration of the line integral along the circumference of a unit circle in 2D, as required by Eq. (26), was carried out by standard Gauss–Legendre quadrature formula employing ten points (Stroud and Secrest,

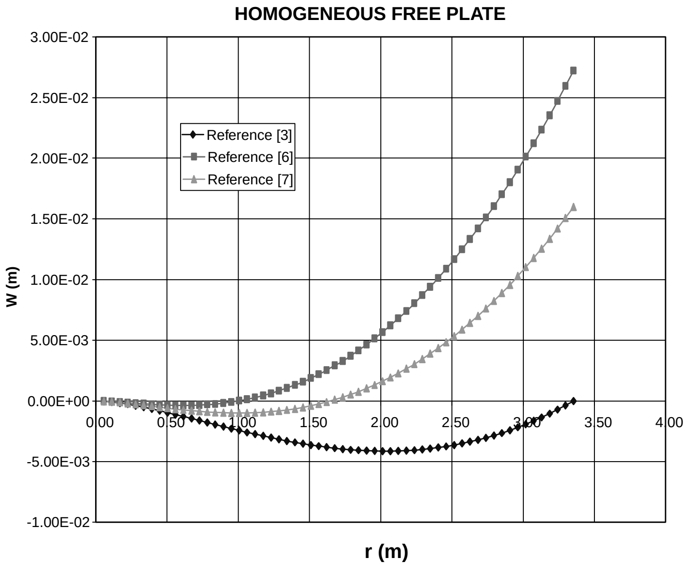

Fig. 3. Homogeneous free plate displacement solution W for a point load and under different boundary conditions versus radial distance r.

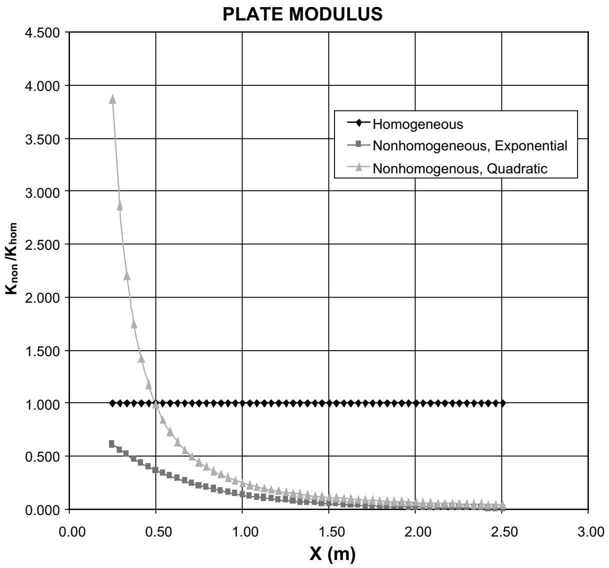

Fig. 4. Normalized elastic plate modulus K/D versus distance x for the non-homogeneous plates defined by the exponential and quadratic conformal mappings.

1966). It is worth noting that similar line integrals appear in the displacement solution for problems exhibiting anisotropy in 2D elastodynamics (e.g., Wang and Achenbach, 1994). These solutions have also been integrated numerically with good accuracy by using the aforementioned Gauss–Legendre formulas.

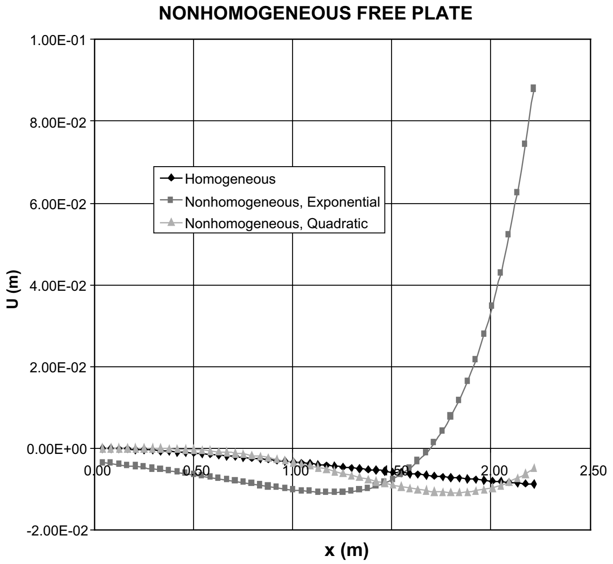

Fig. 5. Non-homogeneous free plate displacement solution W for a point load versus radial distance r.

Finally, Fig. 6 plots the normalized displacement for the non-homogeneous plates supported on an elastic subgrade versus dimensionless length parameter x/l_1. We again observe that the displacement curve of the non-homogeneous plate defined by the quadratic mapping follows quite closely that of the equivalent homogeneous plate, up to a dimensionless distance from source of about 2.5. Past that, the solution starts to exhibit oscillatory behavior. The non-homogeneous plate defined by the exponential mapping, on the

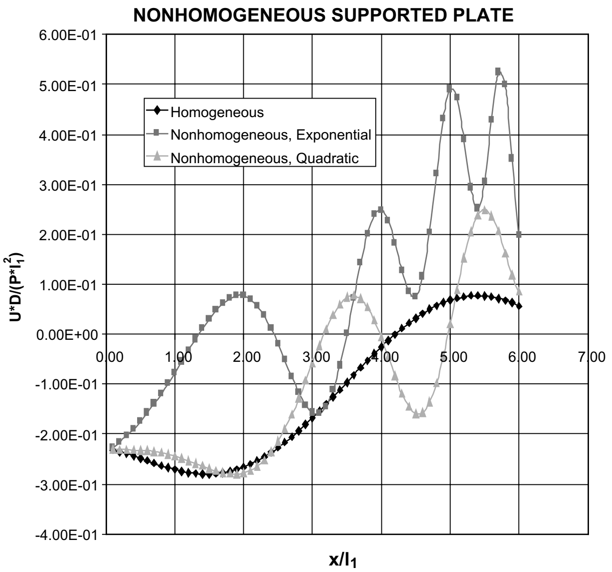

Fig. 6. Non-homogeneous plate on elastic subgrade normalized displacement solution versus normalized distance.

other hand, exhibits oscillatory behavior from the start and, in terms of magnitude, deflects twice as much as the previous plate. The reason for this is again the rapidly diminishing rigidity of the plate with increasing distance from its center.

5. Conclusions

A procedure based on conformal mapping coupled with the Radon transformation is developed in this work for determining fundamental solutions for a class of problems involving the non-homogeneous, biharmonic differential operator. These solutions correspond to plates with position-dependent stiffness resting on an elastic sub-grade. The exact nature of the plate stiffness variability is determined by the particular conformal mapping used, and as examples we mention exponential variation as well as power-law variation with respect to distance from the source where the point load is applied. In the future, it is possible to do a systematic study of the non-homogeneous biharmonic equation, given the wide range of applications it has in the field of structural mechanics. Finally, these functions can be used as kernels in BEM solutions of boundary value problems in order to validate finite difference and finite element numerical approaches for problems involving various types of plates.

Acknowledgement

The first two authors would like to acknowledge the financial support provided under NATO grant no. EST.CLG.977774.

Appendix A. Conformal mapping

If a conformal mapping is introduced whose Jacobian is proportional to modulus $K(r)$, r being the radial distance in the reference coordinate system, the relevant governing equation reduces to a constant coefficient form that can often be solved analytically. These types of mappings are defined by either

$$Z = X + iY = f(z) = f(x + iy) \tag{A.1}$$

in Cartesian coordinates or by

$$Z = R\exp(i\Theta) = f(r\exp(i\theta)) \tag{A.2}$$

in polar coordinates. Consider for instance the following power-type mapping

$$Z = Az^n \tag{A.3}$$

with A, n constants, which has a Jacobian of the form

$$J(r) = (An)^2 r^{2(n-1)} \tag{A.4}$$

This leads to a variable coefficient expression

$$K(r) = K_0 r^{2(n-1)} \tag{A.5}$$

where K_0 is a constant of proportionality. Note that n does not need to be an integer, but must be greater than zero if the material properties are to remain finite at the origin. Finally, this particular transformation produces a relationship between the original and mapped coordinates in the form

$$R = r^n; \quad \Theta = n\theta \tag{A.6}$$

$$|\boldsymbol{R} - \boldsymbol{R}_0| = \left\{ r^{2n} + r_0^{2n} - 2r^n r_0^n \cos\left(n[\theta - \theta_0]\right) \right\}^{1/2} \tag{A.7}$$

More examples of these mappings can be found in Shaw and Gipson (1995) and Shaw and Manolis (2000), Sapountzakis and Katsikadelis (1992) and Faruque and Zaman (1991).

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